Parameter fluctuations in coupled chaotic systems

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We study the effect of parameter fluctuations on synchronization of a coupled chaotic system. The fluctuations to the parameter can be random or it can be a periodic modulation. For random fluctuations we introduce a new quantity, the *rate of fluctuations*, apart from the statistical features of the fluctuations. Fluctuation rate in our study refers to the number of random modifications to parameters occurring in unit time. With a periodic modulation, the fluctuation rates can be the frequency of the modulating term. It is found that with high fluctuation rates the synchronization is stable irrespective of the statistical or mathematical features of fluctuation. It was also found that the low fluctuation rates can destroy synchronization even with a small amplitude. We analytically explain the observed phenomenon using the dynamical equations and numerically verified with a coupled system of Rossler attractor as an example. We also numerically quantify the relation between synchronization error and fluctuation rates.

I. INTRODUCTION

Synchronization of coupled chaotic systems has generated a lot of research activities over the last several years. Synchronized behavior has been studied extensively in physical, chemical and biological systems.Different types of synchronization such as complete, generalized, lag and phase synchrony are described in literature. One of the methods by which the synchronization of chaotic systems is achieved is by coupling two identical systems, which may be unidirectional or bidirectional[1, 2, 3, 6, 9, 10, 11, 12, 13]. Synchronization in arrays of coupled laser systems has also been investigated under various coupling schemes[9, 10, 11, 12, 13]. Complete synchronization of identical chaotic systems is also of considerable interest because of its applications in secure communication[12, 13]. By identical systems we mean a set of systems whose parameters are exactly equal. It is found that the synchronization is not robust when there is a small but finite mismatch of the parameters of the systems[6, 7, 8]. In fact the effect of phase mismatch and a finite constant frequency detuning in a bidirectionally coupled Duffing oscillators is to destroy the synchronization altogether[14].

In the present paper we address a different issue that is relevant in many practical physical systems. We study the effect of random fluctuations in the parameters of the system on the synchronization properties. Such a study is relevant and important since in a real physical system the parameters can often fluctuate randomly either due to some internal instabilities or due to some external perturbations.

The paper is organized as follows; in section II. we consider random perturbations to one of the parameter that characterizes the synchronization. We present the criteria for the robustness of synchronization. In section III. the numerical results on two coupled Rossler systems with randomly fluctuating parameters are presented. Section IV contains the discussions of the result.

II. PARAMETER FLUCTUATIONS IN COUPLED SYSTEMS

In this section we consider two identical dynamical systems which are coupled together. To study the effect of fluctuations it is essential to identify one parameter whose mismatch is most effective in destroying synchronization. We denote this parameter as p. Then the equations for the coupled systems are given by.

$$\dot{X}_1 = f_1(p_1, X_1) + Cf(X_2 - X_1) \qquad (1)$$
$$\dot{X}_2 = f_1(p_2, X_2) + Cf(X_1 - X_2)$$

Here C is the coupling constant. In reality it is difficult if not impossible to construct identical systems except in numerical simulations. This can also be due to the fact that the parameters could be fluctuating in time with a time scale of their own. We incorporate this by writing the parameter as

$$p_1 = p_0 + \xi_{1t} \qquad (2)$$
$$p_2 = p_0 + \xi_{2t},$$

where, ξ_{1t} and ξ_{2t} are two delta correlated zero mean Gaussian random variables. A measure of the amplitude of fluctuations, we define $\widetilde{\Delta p}$, as

$$\widetilde{\Delta p} = \langle | \delta p(t) | \rangle_t, \qquad (3)$$

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where, $\delta p(t) = p_1(t) - p_2(t)$ and $\langle...\rangle_t$ denotes time average.

To study the effect of time scales of parameter fluctuation, we define the fluctuation rate $\phi =$ *number of perturbuations/unit time*. Different fluctuation rates can be achieved numerically by modifying the parameter as in Eqn2 only in certain chosen time steps. Rest of the time the value of the parameter remains constant at the modified value. The Error in synchrony is studied varying ϕ. The effect of time scales has not been studied in literature and our results indicate that it is highly significant in determining the quality of synchronization.

We also considered a periodic modulation of the parameters, where the Eqn. 2 can be replaced by

$$
\begin{aligned}
p_1 &= p_0 + a \sin ft \\
p_2 &= p_0 - a \sin ft
\end{aligned}
\tag{4}
$$

where f is the frequency and a is the amplitude of modulation. By choosing an appropriate value for a and by changing f the quality of synchronization for various modulating frequencies can be studied.

III. NUMERICAL EXAMPLES

Coupled Rossler oscillators are a well known for numerical studies in synchronization. Consider a system of bidirectionally coupled Rossler oscillators. The coupled equation can be written as.

$$
\begin{aligned}
\dot{x_1} &= -y_1 - z_1 + c(x_2 - x_1) \\
\dot{y_1} &= x_1 + p_1 y_1 \\
\dot{z_1} &= 0.2 + z_1(x_1 - 10) \\
\dot{x_2} &= -y_2 - z_2 + c(x_1 - x_2) \\
\dot{y_2} &= x_2 + p_2 y_2 \\
\dot{z_2} &= 0.2 + z_2(x_2 - 10)
\end{aligned}
\tag{5}
$$

Here, the coupling strength $c = 0.15$, and $p_0 = 0.18$. Though the coupling strength can also affect synchronization, we chose a value that is best suited for illustrating the concepts. Also the value of $\widetilde{\Delta p}$ was fixed to be 0.05 for all fluctuation rates.

Fig.1 shows the synchronization plot in the presence of parameter fluctuations. It can be seen that the synchronization is robust. Fig.2 shows that the coupled systems posses a parameter mismatch at any instant of evolution of the system. Also at times the instantaneous mismatches can be compared to the value of the average value parameter value itself. With the same value of $\widetilde{\Delta p}$ the synchronization is destroyed with a lower fluctuation rate as shown in Fig.3.

We have studied the relationship between the synchronization error and the fluctuation rate. To quantify
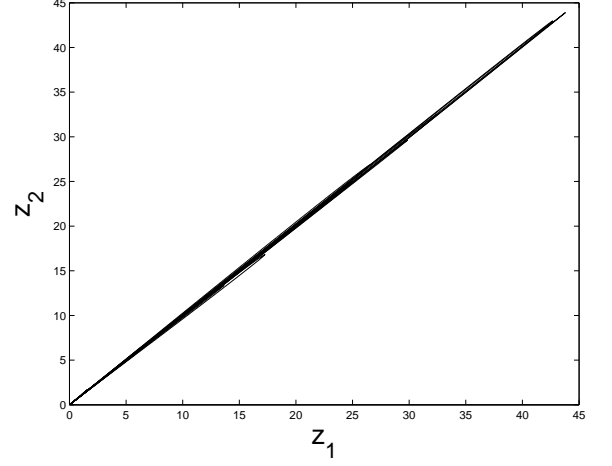

FIG. 1: Synchronization is maintained in the presence of parameter fluctuations. $\phi = 1000$ and $\widetilde{\Delta p} = 0.05$
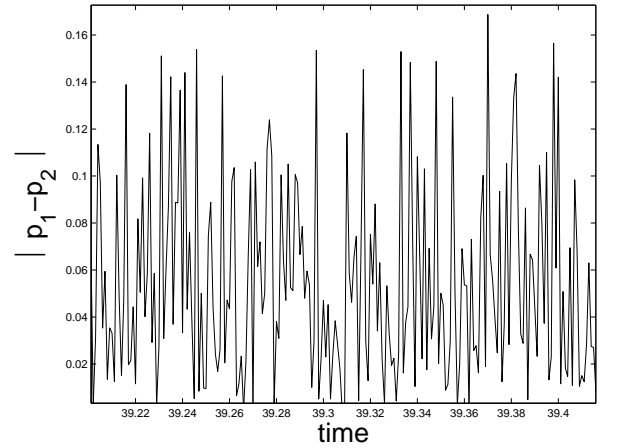

FIG. 2: Synchronized system also posses a zero mean parameter mismatch.

the synchronization error we used the similarity function given by

$$
S^2(\tau) = \frac{\langle [x(t+\tau) - x(t)]^2 \rangle}{[\langle x^2(t) \rangle \langle x^2(t) \rangle]^{\frac{1}{2}}}.
\tag{6}
$$

Here τ is set to zero, which gives $S(0)$, the error in synchrony. Fig. 4 Shows the plot of $S(0)$ vs. fluctuation rate. It can be seen the error diminishes rapidly with the increase in the fluctuation rate.

A similar behavior was also seen with the periodic modulation of parameters. Fig. 5 shows that the synchronization error decreases with increase in the modulating frequency. This clearly suggests that the fluctuation frequency or rate is more important than the nature of fluctuations.
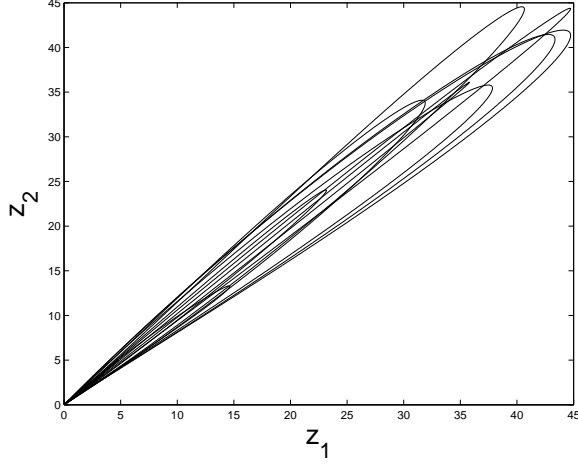
FIG. 3: Synchronization is destroyed in the presence of parameter fluctuations with low fluctuation rates. Fluctuation rate $\phi = 25$ and $\widetilde{\Delta p} = 0.05$

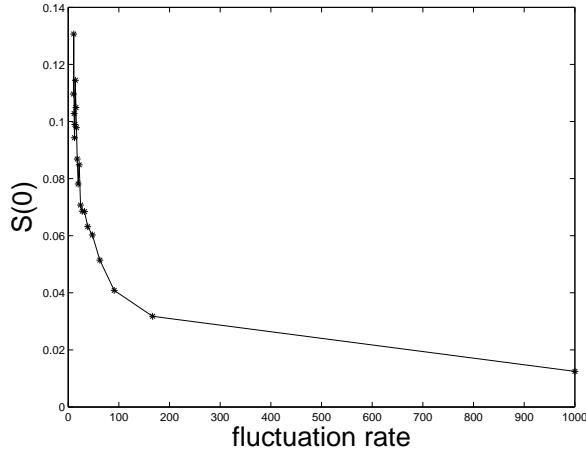

FIG. 4: The synchronization error decreases with the increase in the fluctuation rate.

IV. DISCUSSION

The robustness of synchronization with high fluctuation rates and destruction of synchronization with low fluctuation rates can be explained analytically as follows. With Eqn.1 we can write an equation for the rate of separation $X_1 - X_2$ of the trajectories as,

$$\frac{d(X_1 - X_2)}{dt} = \dot{X}_1 - \dot{X}_2 = M(p_1, p_2, X_1, X_2), \quad (7)$$

$M(p_1, p_2, X_1, X_2)$ is a function of the dynamical variables and the parameters of the coupled systems. This can be written as the sum of two terms,

$$M(X_1, X_2) = S(X_1, X_2) + E(X_1, X_2). \quad (8)$$

This comes from the fact that for a function $\Omega(p, X)$, we can write for small Δp and neglecting its higher pow-
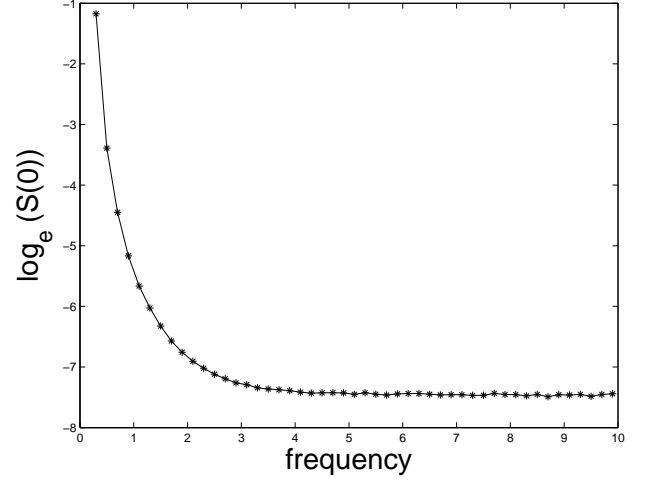

FIG. 5: The synchronization error decreases with the increase the modulating frequency.

ers or if the higher derivatives of Ω w.r.t p is zero,

$$\Omega(p + \Delta p, X) = \Omega(p, X) + \Delta p \frac{\partial \Omega(p, X)}{\partial p}. \quad (9)$$

This can be valid for functions in the dynamical equations of a chaotic system if the parameter values are not near a bifurcation point. Using this relation, with $p_1 = p_0 + \Delta p_1$ and $p_2 = p_0 + \Delta p_2$,

$$S(p_0, X_1, X_2) = F_1(p_0, X_1, X_2) - F_2(p_0, X_1, X_2)$$

and

$$E(p_1, p_2, X_1, X_2) = \Delta p_1 \frac{\partial F_1(p_1, p_2, X_1, X_2)}{\partial p} \Big|_{p_1=p_0}$$
$$- \Delta p_2 \frac{\partial F_2(p_1, p_2, X_1, X_2)}{\partial p} \Big|_{p_2=p_0}.$$

with $\Delta p_1 = \xi_{1t}$ and $\Delta p_2 = \xi_{1t}$.

Here $S(p_0, X_1, X_2)$ represents the quantity which offers a stable synchronization manifold, that is, when $S(p_1, p_2, X_1, X_2)$ alone in the right hand side of the separation equation, coupled systems synchronize as $t \longrightarrow \infty$. The conditions for such a synchronization is widely discussed in literature [15]. The term $E(p_1, p_2, X_1, X_2)$ represents the effect of the parameter mismatch. Coupled systems can synchronize if the overall effect of this term is zero as $t \longrightarrow \infty$. One possible way for this is when $E(p_1, p_2, X_1, X_2, C)$ is of the form

$$E(p_1, p_2, X_1, X_2, C) = \sum_i \rho(t)_i x_i(t) \quad (10)$$

where $\rho_i(t)$ is the the fluctuation term and $x(t)$ is the phase space variables of the coupled system. Here

the equation vanishes because the $\rho(t)$'s are zero mean rapidly fluctuating quantities and $x(t)$'s are the phase space variables that evolve slowly when compared to the the rapid fluctuations of the parameter. Thus $x(t)$'s can be assumed to be constant, in the time required for the fluctuations get summed to zero. This also explains why the synchronization is destroyed when the fluctuation rate is low. With a low fluctuation rate the $E(p_1, p_2, X_1, X_2, C)$ cannot be summed to zero every time since the phase space evolution time is comparable to the interval where a fixed parameter mismatch persists. Thus with a lower fluctuation rate the system always get time to respond to the parameter mismatch before it being canceled out. Also with a slowly varying parameter mismatch, a definite state of phase synchrony is also not attained and the system remains in a transient state through out the evolution in the phase space.

In the present example, it can be seen that the quantity $E(p_1, p_2, X_1, X_2, C)$ can be expressed as,

$$E(p_1, p_2, X_1, X_2, C) \; = \; \xi_{1t} y_1 - \xi_{2t} y_2 \qquad (11)$$

because the fluctuating terms appear only in the equation of $\dot{y}$ only. Similar studies were done to the other parameters of the coupled systems as well, which gave similar result. Apart from gaussian random fluctuations, we studied perturbations with a uniform distribution. The results were qualitatively the same as for the gaussian perturbations which suggests that the most important quantity that determines the stability of synchronization is the fluctuation rates.

V. CONCLUSIONS

In this paper we studied the effect of parameter fluctuations on the synchronization of coupled chaotic systems. We investigated random parameter fluctuation and also a periodic modulation to the parameter. It was found that the most significant entity that determines the quality of synchronization is the fluctuation rates that we have defined or the frequency of fluctuation. Our study also show that the timescales with which the parameter fluctuates is more significant than the statistical or mathematical features of the fluctuations.

The effect of noise on synchronization has been studied in the past. Noise affects synchronization in different manner in various situations. In most of the cases noise destroys synchronization or make it unfit for the secure communication purposes that we have cited in the introduction [16, 17]. Also there are cases where synchronization is robust to noise[18] or even induce synchronization [19].

The effect of noise and parameter fluctuations are different. Noise induces perturbations to the phase space variables that decay while the system evolves. In a case where the parameter fluctuates, the resultant perturbations do not die out with the evolution of the system. It remains the same until it is corrected manually or the fluctuation modifies the parameter to a new value. Due to this reason, the fluctuation rate plays an important role in determining the stability of synchronization in coupled chaotic systems. Parameter fluctuations may also have much higher significance in coupled arrays of nonlinear oscillators, especially in biological systems which exhibit synchronized behavior. Though we have not included these in our present paper, we hope that our studies will be a motivation in this direction.

VI. ACKNOWLEDGMENTS

[1] T Yamada and H Fujisaka 1983 *Prog. Theor. Phys.* **70** 1240 bibitempeco1 L M Pecora and T L Carroll 1990 *Phys. Rev. Lett.* **64** 821
[2] T L Carroll and L M Pecora 1991 *IEEE Trans. Circuits Syst.* **38** 453
[3] T L Carroll and L M Pecora 1995 *IEEE Trans. Circuits Syst.* II **40** 646
[4] P Richard, B M Bakker, B Teusink, K Van Dam and H V Westerho 1996 *Eur. J. Biochem.* **235** 238
[5] L Glass 2001 *Nature* **410** 277
[6] A Pikowsky, R Rosenblum and J Kurth 2001 *Syncronization: A universal concept in nonlinear sciences* (Cambridge: Cambridge University Press)
[7] M G Rosenblum, A S Pikovsky and J Kurths 1995 *Phys. Rev. Lett.* **76** 1804
[8] M G Rosenblum, A S Pikovsky and J Kurths 1997 *Phys. Rev. Lett.* **78** 4193
[9] John R. Terry, K. Scott Thornburg, Jr., David J. DeShazer, Gregory D. VanWiggeren, Shiqun Zhu, Peter Ashwin, and Rajarshi Roy 1999 *Phys. Rev. E* **59** 4036
[10] Gregory D. VanWiggeren and Rajarshi Roy 1998 *Science* **279** 1198 3547
[11] P Colet and R Roy 1994 *Opt. Lett.* **19** 2056
[12] Gregory D. VanWiggeren and Rajarshi Roy *Phys. Rev. Lett.* **81** 1998
[13] V Bindu and V M Nandakumaran 2002 *J. Opt. A: Pure Appl. Opt.* **4** 115
[14] H W Yin and J H Dai and H J Zhang 1998 *Phys. Rev.* E **58** 5683
[15] Louis M. Pecora, Thomas L. Carroll, Gregg A. Johnson, and Douglas J. Mar 1997 **7** *Chaos* 520
[16] P Ashwin, J Buescu, and I Stewart 1994 *Phys. Lett.* A **193** 126
[17] D J Gauthier and J C Bienfang 1996 *Phys. Rev. Lett.* **77**

1751

[18] Thomas L. Caroll 2005 *Chaos* **15** 13901

[19] Ral Toral, Claudio R. Mirasso, Emilio Hernndez-Garca and Oreste Piro 2001 *Chaos* **11** 665